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CORPORATE OFFICE :
"SANKALP", CP-6, Indra Vihar, Kota (Raj.), India, 324005

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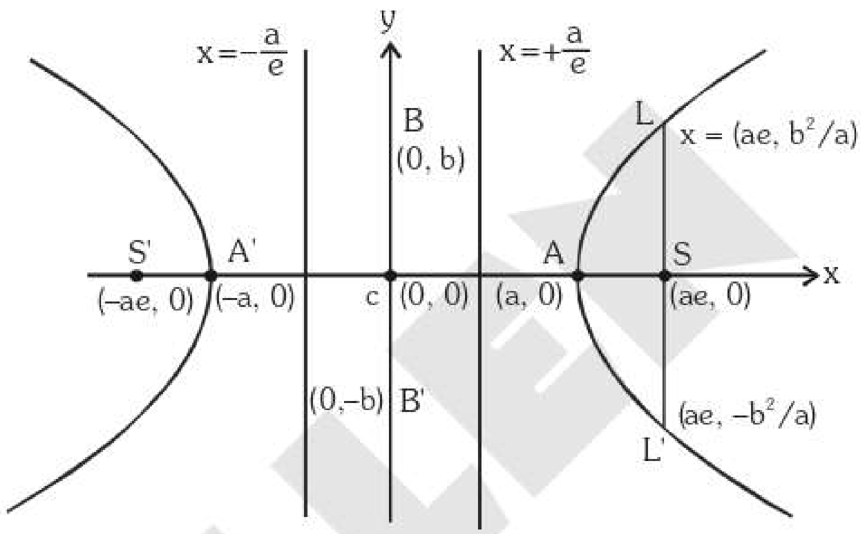
HYPERBOLA

The **Hyperbola** is a conic whose eccentricity is greater than unity. $(e > 1)$.

1. STANDARD EQUATION & DEFINITION(S) :



Standard equation of the hyperbola is $\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

where $b^2 = a^2(e^2 - 1)$

or $a^2e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \frac{b^2}{a^2} = 1 + \left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

(a) Foci :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(b) Equations of directrices :

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

(c) Vertices :

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

(d) Latus rectum :

(i) Equation : $x = \pm ae$

(ii) Length $= \frac{2b^2}{a} = \frac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a\,(e^2 - 1)$

$= 2e$(distance from focus to directrix)

(iii) Ends : $\left(ae, \frac{b^2}{a}\right), \left(ae, \frac{-b^2}{a}\right); \left(-ae, \frac{b^2}{a}\right), \left(-ae, \frac{-b^2}{a}\right)$

(e) (i) Transverse Axis :

The line segment A′A of length 2a in which the foci S′ & S both lie is called the Transverse Axis of the Hyperbola.

(ii) Conjugate Axis :

The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the Conjugate Axis of the Hyperbola.

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the Principal axis of the hyperbola.

(f) Focal Property :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\big|\,|PS| - |PS'|\,\big| = 2a$. The distance SS′ = focal length.

(g) Focal distance :

Distance of any point P(x, y) on hyperbola from foci $PS = ex - a$ & $PS' = ex + a$.

2. CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axis of one hyperbola are respectively the conjugate & the transverse axis of the other are called **Conjugate Hyperbolas** of each other. eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ &

$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each other.

Note that :

(i) If e_1 & e_2 are the eccentricities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

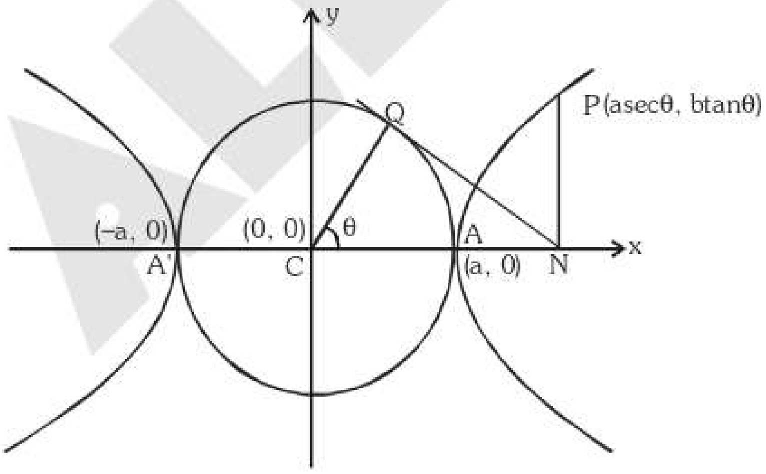
(ii) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(iii) Two hyperbolas are said to be similar if they have the same eccentricity.

3. RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral Hyperbola**. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of it's latus rectum is equal to it's transverse or conjugate axis.

4. AUXILIARY CIRCLE :



A circle drawn with centre C & transverse axis as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P & Q are called the **"Corresponding Points"** on the hyperbola & the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola. $(0 \le \theta < 2\pi)$.

Parametric Equation :

The equations $x = a \sec\theta$ & $y = b \tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter.

5. POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is positive , zero or negative according as the point (x_1, y_1) lies within , upon or outside the curve.

6. LINE AND A HYPERBOLA :

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 > = < a^2m^2 - b^2$.

Equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining its two points $P(\alpha)$ & $Q(\beta)$ is $\frac{x}{a}\cos\frac{\alpha - \beta}{2} - \frac{y}{b}\sin\frac{\alpha + \beta}{2} = \cos\frac{\alpha + \beta}{2}$

7. TANGENT TO THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(a) Point form : Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Note : In general two tangents can be drawn from an external point (x_1y_1) to the hyperbola and they are $y - y_1 = m_1(x - x_1)$ & $y - y_1 = m_2(x - x_2)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$. If $D < 0$, then no tangent can be drawn from (x_1y_1) to the hyperbola.

(b) Slope form : The equation of tangents of slope m to the given hyperbola is $y = mx \pm\sqrt{a^2m^2 - b^2}$. Point of contact are

$$\left(\frac{\pm a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 - b^2}}\right)$$

Note that there are two parallel tangents having the same slope m.

(c) Parametric form : Equation of the tangent to the given hyperbola at the point (a sec θ , b tan θ) is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1.$$

Note : Point of intersection of the tangents at θ_1 & θ_2 is

$$x=a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)},\quad y=b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$$

8. NORMAL TO THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(a) Point form : Equation of the normal to the given hyperbola at the point P (x_1 , y_1) on it is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2=a^2e^2$.

(b) Slope form : The equation of normal of slope m to the given hyperbola is $y=mx\mp\frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$ foot of normal are

$$\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}},\mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$$

(c) Parametric form : The equation of the normal at the point P (a sec θ , b tan θ) to the given hyperbola is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2=a^2e^2.$$

9. DIRECTOR CIRCLE :

The locus of the intersection of tangents which are at right angles is known as the **Director Circle** of the hyperbola. The equation to the director circle is : $x^2+y^2=a^2-b^2$.

If $b^2<a^2$ this circle is real ; if $b^2=a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

10. CHORD OF CONTACT :

If PA and PB be the tangents from point $P(x_1,y_1)$ to the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then the equation of the chord of contact AB is

$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ or $T = 0$ at (x_1,y_1)

11. PAIR OR TANGENTS :

If $P(x_1,y_1)$ be any point lies outside the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and a pair of tangents PA, PB can be drawn to it from P. Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$, $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

i.e. $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$

12. EQUATION OF CHORD WITH MID POINT (x_1,y_1) :

The equation of the chord of the ellipse $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

whose mid-point be (x_1,y_1) is $T = S_1$

where $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$, $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

i.e. $\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)$

13. ASYMPTOTES :

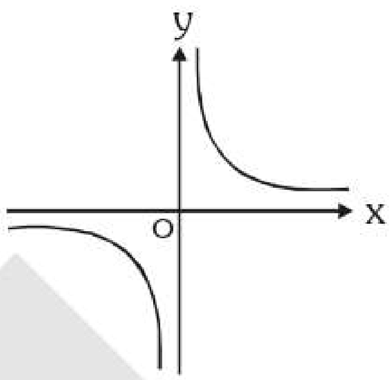
Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the **Asymptote of the Hyperbola**.

Combined equation of asymptotes of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ will be

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

14. RECTANGULAR HYPERBOLA :

Rectangular hyperbola referred to its asymptotes as axis of coordinates.



(a) Equation is $xy = c^2$ with parametric representation $x = ct,\ y = c/t,\ t \in R - \{0\}$.

(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1t_2y = c(t_1 + t_2)$ with slope $m = \frac{-1}{t_1t_2}$

(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$

& at $P(t)$ is $\frac{x}{t}+ty=2c$.

(d) Equation of normal is $y-\frac{c}{t}=t^2(x-ct)$

(e) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

15. IMPORTANT HIGHLIGHTS :

(i) The tangent and normal at any point of a hyperbola bisect the angle between the focal radii.

(ii) **Reflection property of the hyperbola :** An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.